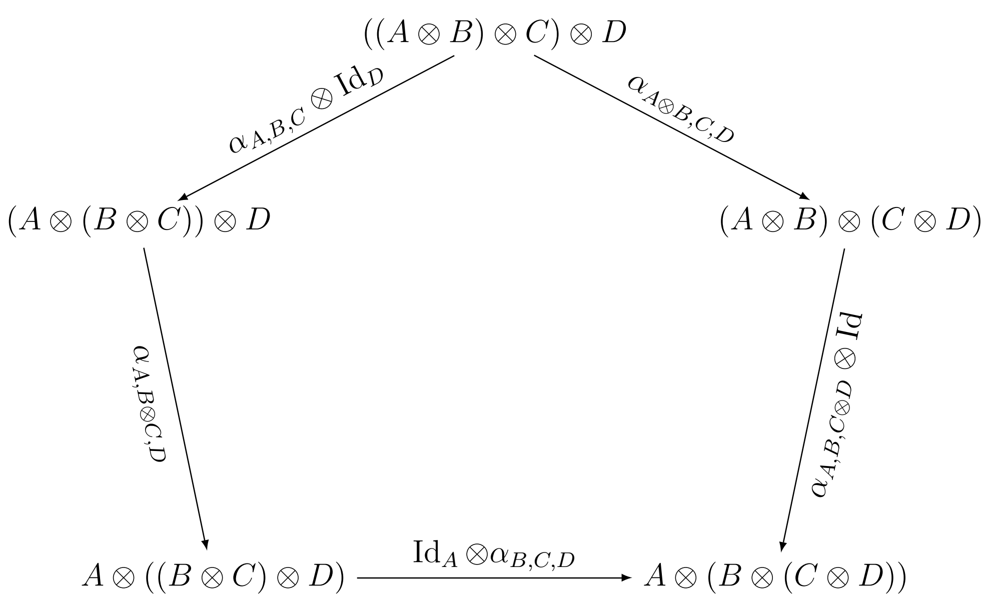
TOP-ENRICHED CATEGORIES

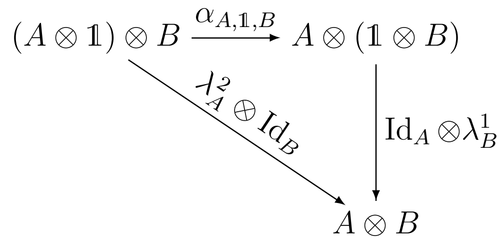
SITAN CHEN

1. Monoidal Categories

Definition 1. *A* monoidal category *is a category with the additional data: i) "associative multiplication of categories", i.e. a map for combining objects, $\cdot \otimes \cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, such that for all $A, B, C \in \mathcal{C}_{obj}$, there exists a natural isomoprhism $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, ii) "identity", i.e. some object $\mathbb{1}$ such that for all $A \in \mathcal{C}_{obj}$, there exist natural isomorphisms $\lambda^1_A : 1 \otimes A \to A$ and $\lambda^2_A : A \otimes 1 \to A$.*

Because we only stipulate associativity up to isomorphism, we have to make sure that all expansions of a products are the same, i.e. that we can essentially imagine our $\alpha_{A,B,C}$, λ^1_A, and λ^2_A are equalities so that strings of the form $A_1 \otimes A_2 \otimes \cdots A_n$ are well-defined up to isomorphism. By a result called Mac Lane's coherence theorem (we omit the proof), it suffices to show that i) the two different expansions from $((A \otimes B) \otimes C) \otimes D$ to $A \otimes (B \otimes (C \otimes D))$ in the first of the following diagrams are the same (convince yourself that there are only two such expansions), ii) $\alpha_{A,\mathbb{1},B}$ and λ^2_A,λ^1_B "play nicely" in the sense that we can expand $(A \otimes \mathbb{1}) \otimes B$ into $A \otimes B$ either directly by $\lambda^2_A \otimes \mathrm{Id}_B$ or by $(\mathrm{Id}_A \otimes \lambda^1_B) \circ \alpha A, \mathbb{1}, B$, as shown in the subsequent diagram.





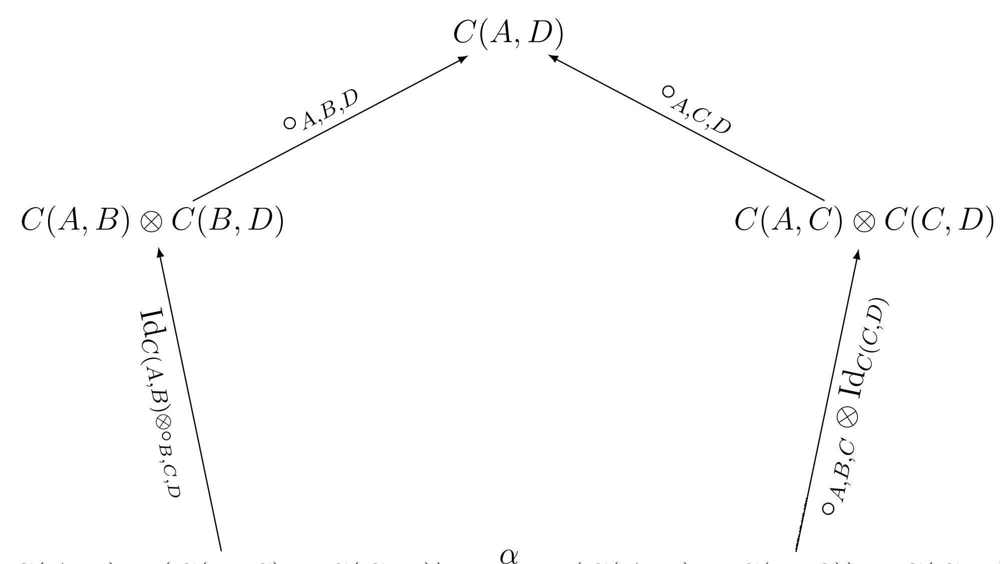
Example 1. *The category of finite sets equipped with $\otimes$ the Cartesian product and $\mathbb{1}$ the single-point set is a monoidal category. The category of R-modules over a commutative ring equipped with $\otimes$ the tensor product and $\mathbb{1}$ the ring R itself is a monoidal category.*

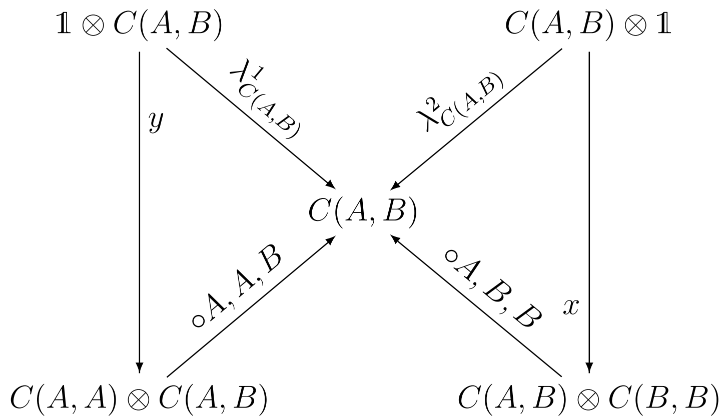
Example 2. *The category* **Top** *of topological spaces equipped with $\otimes$ the Cartesian product and $\mathbb{1}$ the topological space with underlying set $\{*\}$ is a monoidal category.*

2. Enriched Categories

Definition 2. *A category $\mathcal{C}$ is enriched over a monoidal category $\mathcal{M}$ is the data of i) a set of objects, ii) for each pair of objects $A, B \in \mathcal{C}_{obj}$, there is an object in $\mathcal{M}_{obj}$, denoted by $C(A, B)$, which we think of as the generalization of a "morphism" from A to B, iii) for each triple of objects (A, B, C) in $\mathcal{C}_{obj}$, there is a morphism $\circ_{A,B,C} : C(A, B) \otimes C(B, C) \to C(A, C)$ given by the product $\mathcal{M}$ is equipped with, which we think of as a generalization of a "composition law," iv) for each object in $\mathcal{C}_{obj}$ there is an "identity morphism," namely a morphism $i_A : \mathbb{1} \in \mathcal{M}_{obj} \to C(A, A)$.*

We want to make sure the associativity (up to isomorphism) property of the monoidal category, the composition law, and the identity morphism, again, all "play nicely" with each other, so we get two familiar looking coherence conditions:





Example 3. *But recall that **Top** is a monoidal category so that we have a notion of **Top**-enriched categories $\mathcal{C}$. The category of compactly generated Hausdorff spaces, call it* **CGHaus** is a **Top**-enriched category.

Remark 1. *Unfortunately,* **Top** itself is not a **Top**-enriched category because the composition map $C(X,Y) \times C(Y,Z) \to C(X,Z)$ need not be continuous in the compact-open topology.

Proof. For any two spaces X, Y, the space of continuous functions $C(X,Y)$ is also a space when endowed with the *compact-open topology*: the set of all functions sending a compact subset $K \subset X$ into an open subset $U \subset Y$ forms a sub-basis for the topology. We first need to verify that for every triple of spaces X, Y, Z, there is a morphism $\circ_{X,Y,Z} : C(X,Y) \otimes C(Y,Z) \to C(X,Z)$. But this can just be given by composition: for any $f \in C(X,Y)$ and $g \in C(Y,Z)$, send $f \otimes g$ to $g \circ f$, and because X, Y, Z are compactly generated Hausdorff, the composition map is continuous.

Next, for each space $X \in \mathbf{CGHaus}_{obj}$, we can pick the identity morphism to simply be the identity map in $C(X,X)$. It is straightforward to verify that our two coherence diagrams commute. □

Proposition 1. *For objects A, B, C in* **CGHaus**, equipped with the structure of a **Top**-enriched category $\mathcal{C}$, there is a natural homeomorphism $\phi : C(A \otimes B, C) \cong C(A, C(B,C))$.

Remark 2. *In computer science, specifically functional programming, this equivalence is known as "currying."*

Proof. Begin with an f in $C(A, C(B,C))$ and send this to g such that $g(x,y) = f(x)(y)$. In the other direction, begin with a g in $C(A \otimes B, C)$ and send this to an f such that $f(x)$ sends y to $g(x \otimes y)$. We want to show that this map is continuous in both directions.

In the first direction, it suffices to show that the elements in the sub-basis of the compact-open topology of $C(A \otimes B, C)$ are sent to open sets in $C(A \otimes B, C)$; in fact, it suffices to show that for each $K \subset A$ compact, $L \subset B$ compact, and $U \subset C$ open, the open element $(K, (L,U))$ denoting the set of all maps that take $K \subset A$ inside the set of all maps from L into U, is taken to an open set in $C(A \otimes B, C)$, namely the set of all maps $(K \otimes L, U)$, and because the product of two compact sets is compact, we are done in this direction.

In the other direction, begin with a sub-basis element (K,U) of $C(A \otimes B, C)$ where K is compact in $A \times B$ and U is open in C. The basic idea is that we want to approximate our compact set K, which certainly is not a simple cartesian product of compact sets in A and

B, by finitely many compact boxes. Specifically, we want to show that for any $f \in \phi(K,U)$, there are compact subsets K_1,...,K_n of A and compact subsets $L_1, ..., L_n$ of B such that $f \in \cap_{i=1}^n (L_i, (K_i, U)) \subset \phi(K,U)$.

Denote the function corresponding to f to be $f_0 \in (K,U)$. For each $(x,y) \in K$, choose a neighborhood $U_0^{x,y}$ of x in K_x and a neighborhood $V_0^{x,y}$ of y in K_y, where K_x and K_y denote the projections of $A \times B$ onto A and B, respectively, and pick them such that $f_0(U_0^{x,y} \times V_0^{x,y}) \subset W$, which we can do by continuity of f_0. Now we want to pick closed neighborhoods of x contained inside these larger neighborhoods: for each $x \in K_x$ and each $y \in K_y$, choose a neighborhood $U^{x,y}$ and $V^{x,y}$, respectively, such that their closures $K^{x,y}$ and $L^{x,y}$ are in $U_0^{x,y}$ and $V_0^{x,y}$. By compactness of K, we can pick a finite open subcovering of J by choosing from our covers $(U^{x,y} \times V^{x,y}) capK$, and denote the covers by $(U^i \times V^i) \cap J$ for $i = 1, ..., n$. For convenience, call the $U_0^{x,y}$ ($V_0^{x,y}$) and $K^{x,y}$ ($L^{x,y}$) corresponding to each U^i (V^i) by U_0^i (V_0^i) and K^i (L^i), respectively.

Then note that $f(L_i)(K_i) = f_0(K^i \times L^i) \subset f_0(U_0^i \times V_0^i) \subset W$, and because K_i and L_i are closed subsets of compact sets, namely K_x and K_y, respectively, they are compact. As a result, f lies inside $\cap_{i=1}^n (L_i, (K_i, U))$. We want to show that

$$\cap_{i=1}^n (L_i, (K_i, U)) \subset \phi(K,U),$$

so take some g on the left and call its preimage in ϕ the function g_0. We know that $g_0(K^i \times L^i) = g(L^i)(K^i) \subset U$ by definition, and because each $(x,y) \in K$ belongs to such a compact box $K^i \times L^i$, $g_0 \in (K,U)$ and thus $g \in \phi(K,U)$ as desired. $\square$