# On the Becker-Gottlieb Transfer and a Proof of the Adams Conjecture

Sitan Chen

May 4, 2014

## 1 Introduction

In this paper, we present an elementary proof of the Adams Conjecture due to Becker and Gottlieb [1] using a generalization of the transfer homomorphism of Kahn and Priddy to fiber bundles with compact manifolds as fibers.

Given a fiber bundle $F \hookrightarrow E \xrightarrow{p} B$, where $F$ is a compact manifold acted on smoothly by a compact Lie group $G$, and where $B$ has the homotopy type of a finite CW complex, the *Becker-Gottlieb transfer* $\tau^*$ is a homomorphism in singular cohomology induced by a map $\tau : B^+ \to E^+$ such that $(p \circ \tau)^*$ is multiplication by the Euler characteristic $\chi(F)$. Here, $B^+$ and $E^+$ are the based spaces given by the disjoint union of $B$ and $E$ with a point.

Once the transfer is constructed, we use it to prove that for a fiber bundle $F \hookrightarrow E \to B$ where $\chi(F) = 1$, $p^* : h^*(B^+) \to h^*(E^+)$ is an injective homomorphism onto a direct factor. This allows us to prove a variant of the splitting principle which we use to reduce proof of the Adams Conjecture from arbitrary vector bundles to $2n$-plane bundles to bundles associated to the principal $G$-bundle for a particular group $G$, and finally to talking about bundles of dimension 2, for which we already have a proof due to Adams.

## 2 Constructing The Transfer Map

### 2.1 An Aside About Fiber Bundles

Our arguments make heavy use of the notion of principal $G$-bundles and associated bundles, whose definitions we believe it will be helpful to provide before diving into the actual proof.

**Definition 1.** *For $G$ a topological group, a* principal $G$-bundle *is a fiber bundle $\pi : E \to X$ equipped with a continuous action $E \times G \to E$ of $G$ on the total space that acts freely and transitively on the fibers of $E$ and such that for all fibers $E_x$, all $e \in E_x$, and all $g \in G$, $eg \in E_x$, i.e. the fibers are preserved.*

**Definition 2.** *Given a principal $G$-bundle $\pi : E \to X$ and a continuous action $\rho : G \to \mathrm{Homeo}(F)$ on some space $F$, the* bundle associated to $\pi$ *is the fiber bundle $\pi_\rho : E \times_\rho F \to X$, where $E \times_\rho F$ is the $E \times F$ modulo the reltaion $(p, f) \sim (pg, \rho(g^{-1})f)$.*

**Remark 1.** *The structure group of the bundle associated to a principal $G$-bundle has structure group $G$.*

## 2.2 The Construction

Specifically, for a principle $G$-bundle $\tilde{\xi} : \tilde{E} \xrightarrow{\tilde{p}} B$, let $\xi$ be the associated bundle with fiber a $G$-manifold $F$. We will construct an map $\tau(\xi) : B^+ \to E^+$ such that:

(i) $\tau$ *is natural*: any bundle map $h : \xi \to \xi'$, where $\xi' : E' \xrightarrow{p'} B'$, the diagram

$$\begin{array}{ccc} B^+ & \xrightarrow{\tau(\xi)} & E^+ \\ \downarrow h & & \downarrow h \\ B'^+ & \xrightarrow{\tau(\xi')} & E'^+ \end{array}$$

commutes

(ii) $\tau$ *respects products*: for any finite CW complex $X$, $\tau(X \times \xi) = 1 \wedge \tau(\xi)$, where $X \times \xi$ denotes the bundle $X \times E \xrightarrow{\mathrm{id} \times p} X \times B$.

(iii) If the base space is the single-point space so that the total space is the fiber, then $p \circ \tau : \{0\}^+ \to \{0\}^+$ has degree $\chi(F)$.

*Proof.* As we're dealing with a principle $G$-bundle $\tilde{\xi}$, it will be helpful to introduce ex-spaces/ex-maps as well as two important examples:

**Definition 3.** *An* ex-space *is the data of spaces $X$ and $B$, a projection map $p : X \to B$ and "cross-section" map $\Delta : B \to X$ such that $p\Delta = 1$. For two such ex-spaces $X$ and $Y$ over the same $B$, denote the fiberwise smash product by $X \wedge_B Y$.*

*An* ex-map *is a map of ex-spaces $X \to Y$ respecting corresponding projection and cross-section maps.*

**Example 1.** *For a vector bundle $\alpha : X_\alpha \xrightarrow{p_\alpha} B$, one-point compactify each of the fibers to get some $\bar{\alpha} : X_{\bar{\alpha}} \xrightarrow{p_{\bar{\alpha}}} B$. This is an ex-space: the cross-section map $\Delta : B \to X_{\bar{\alpha}}$ sends $b \in B$ to the point at infinity in the fiber over $b$.*

**Example 2.** *For a principle $G$-bundle $\tilde{p} : \tilde{E} \to B$ and $G$-space $Y$ with base point $*$ fixed under the action of $G$, the fiber product $\tilde{E} \times_G Y$ is an ex-space over $B$ where the cross-section map $\Delta : B \to \tilde{E} \times_G Y$ is given by $b \mapsto \overline{(\tilde{e}, *)}$, where $\tilde{p}(\tilde{e}) = b$.*

For fiber $F$, the Mostow-Palais Theorem (stated here without proof) tells us that there exists an equivariant embedding $F \hookrightarrow V$ for a $G$-module $V$.

Our construction makes use of the Pontryagin-Thom collapse map, the definition of which we outline below, as well as the notion of a Thom space, which we also recall below. For a space $X$, denote its one-point compactification by $S^X$.

**Definition 4.** *For a (locally compact Hausdorff) space $V$ and an open subset $U \subseteq V$, the* Pontryagin-Thom collapse *is the map $S^V \to S^V/(S^V - U) \simeq S^U$ given by collapsing the complement of $U$ in $S^U$.*

**Definition 5.** *Given a vector bundle $\xi : E \to B$ equipped with an inner product on the fibers, the* disk (sphere) bundle *is the fiber bundle $D(\xi) : D(E) \to B$ ($S(\xi) : S(E) \to B$) with fibers the disks (spheres) of radius 1.*

*The* Thom space of $\xi$ over $B$, *which we denote by $B^\xi$, is then $D(\xi)/S(\xi)$.*

*Equivalently, we can define the Thom space of a vector bundle without an inner product to be the fiberwise one-point compactification of the total space in which the points at infinity of the fibers are identified.*

We want to make use of the Pontryagin-Thom collapse $c$, so we would like an open embedding inside $V$. To this end, consider the normal bundle $\omega : X_\omega \xrightarrow{\omega} F$ of this inclusion; the total space $X_\omega$ is an open tubular neighborhood of $F$ inside $X$, so the Pontryagin-Thom collapse so we get the map

$c : S^V \to F^\omega$, where $F^\omega$ denotes the Thom space of $\omega(i)$ over $F$, where $i$ is the inclusion $F \hookrightarrow V$. In particular, the collapse map sends $S^V$ to the one-point compactification of the total space $X_\omega$, i.e. the Thom space $F^\omega$ by the second definition above.

Now denote the tangent bundle on $F$ by $\tau$ and the trivialization $\tau \oplus \omega \to F \times V$ by $\psi$. We get a $G$-equivariant map

$$\gamma : S^V \xrightarrow{c} F^\omega \hookrightarrow F^{\tau \oplus \omega} \xrightarrow{\psi} F^+ \wedge_B S^V \xrightarrow{\pi} S^V.$$

We now use the construction in our second example to get an ex-map $\gamma' := 1 \wedge \gamma : \tilde{E} \wedge_G S^V \to \tilde{E} \wedge_G (F^+ \wedge_B S^V)$.

Now let $\eta$ be the vector bundle with fiber $B$ associated with $\tilde{E}$. The fibers lie in some $\mathbb{R}^s$, so take $\zeta : X_\zeta \to B$ to be the fiberwise complement of $\eta$.

Recall that we want a stable map $B^+ \twoheadrightarrow \tau(\xi)E^+$, i.e. a map between the Thom spaces of $B \times \mathbb{R}^n$ and $E \times \mathbb{R}^n$. If $\Phi : \eta \oplus \zeta \to B \times \mathbb{R}^s$ is a trivialization, then $B^+ \wedge S^s \simeq B^{\eta \oplus \zeta}$ and $E^+ \wedge S^s \simeq E^{p^*(\eta \oplus \zeta)}$ via $\Phi$ and $p^*(\Phi)$.

To get a map $p_! : B^{\eta \oplus \zeta} \to E^{p^*(\eta \oplus \zeta)}$, the so-called "Gysin map", we smash one more time to get the map

$$\gamma' \wedge_B 1 : (\tilde{E} \wedge_G S^V) \oplus_B X_{\bar{\gamma}} \to \tilde{E} \wedge_G (F^+ \wedge_B S^V) \oplus_B X_{\bar{\gamma}},$$

where $X_{\bar{\gamma}}$ denotes the fiberwise one-point compactification of total space $X_{\bar{\gamma}}$.

We now show that by collapsing $B$ to a point on both sides of this map, we get the desired map $p_!$. We do this by looking at a single fiber on both sides. On the left, $G \times_G S^V$ is a collection of pairs $(g, y)$ such that $(g, y) \sim (1, gy)$ but because only $\infty \in S^V$ is fixed by $G$, $G \times_G S^V$ is merely the one-point compactification of $G$, with point at infinity $(1, \infty)$ identified across fibers. Because we are smashing this with the one-point compactification of a fiber in $X_{\bar{\zeta}}$, we indeed get $B^{\eta \oplus \zeta}$ from collapsing $B$ to a point in the left-hand side of the map $\gamma' \wedge_B 1$.

On the right-hand side, the same reasoning holds, but because $\xi$ is the bundle with fiber $F$ associated with $\tilde{\xi}$, we get that the right-hand side with $B$ collapsed to a point is $E^{p^*(\eta \oplus \zeta)}$, as desired.

Our $\tau(p)$ is thus the (stable) composition

$$(B^+) \wedge S^s \xrightarrow{\Phi^{-1}} B^{\eta \oplus \zeta} \xrightarrow{p_!} E^{p^*(\eta \oplus \zeta)} \xrightarrow{p^*(\Phi)} (E^+) \wedge S^s.$$

It remains to show that our construction does not depend on the choice of $V$: say we had two equivariant embeddings $e : F \to V$ and $e' : F \to V'$

giving $\gamma$ and $\gamma'$. There is an isotopy (homotopy preserving embeddings) $H : F \times I \to V \oplus V'$ from $e$ to $e'$ given simply by $H_t(f) = (1-t)e(f) \oplus te'(f)$. With this $H$, we can construct an equivariant homotopy $K : S^{V\otimes V'} \times I \to (F^+) \wedge S^{V\oplus V'}$ from $\gamma \wedge 1$ to $\gamma \wedge 1$, and following the construction above of $p_!$ given $\gamma$, we conclude that $\tau(p)$ is well-defined up to homotopy.

We can easily verify that the transfer map we've constructed is natural and respects products. In the following discussion, we will prove property iii) by proving the following claim:

**Claim 1.** *$\gamma$ has degree $\chi(F)$.*

*Proof.*

**Case 0.** *$F$ is connected and orientable.*

Pick $U_\tau$ a Thom class for $\tau$ and let $U_\omega$ be the corresponding Thom class for $\omega$ such that for the natural map $d : F^{\tau\oplus\omega} \to F^\tau \wedge F^\omega$, $d^*(U_\tau \wedge U_\omega) = \psi^* \circ \pi^*(v)$ for $v$ a canonical generator of $H^*(S^V) = H^*(S^s)$, where $s$ is the dimension of $V$. We want to show that $c^*i^*\psi^*\pi^*(v) = \chi(F)v$, but by definition of our compatible Thom classes $U_\tau$ and $U_\omega$, it suffices to show that $c^*i^*d^*(U_\tau \wedge U_\omega) = \chi(F)v$

The Thom isomorphism $\Phi$ given by multiplication by $U_\omega$ is a map $\Phi : \tilde{H}^n(F^+) \to \tilde{H}^s(F^\omega)$. Pick a generator $v$ of $\tilde{H}^s(S^s)$ and denote $\mu \in \tilde{H}^n(F^+)$ to be the preimage of $v$ in

$$\tilde{H}^n(F^+) \xrightarrow{\Phi} \tilde{H}^s(F^\omega) \to \tilde{H}^s(S^v) = \tilde{H}^s(S^s).$$

Let $h$ denote the inclusion of $F$ into $F^\tau$. At the level of bundles, we have the diagram

$$\begin{array}{ccc} \omega & \xrightarrow{\ i\ } & \tau \oplus \omega \\ \rho \downarrow & & \downarrow d \\ \{0\} \times \omega & \xrightarrow{h \times 1} & \tau \times \omega \end{array}$$

where $\rho$ is the natural inclusion of $\omega$ into $\{0\} \times \omega$. At the level of Thom spaces, we then have the diagram

$$
\begin{array}{ccc}
F^{\omega} & \xrightarrow{\quad i \quad} & F^{\tau\oplus\omega} \\
\Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle d} \\
F^{+} \wedge F^{\omega} & \xrightarrow{h\wedge 1} & F^{\tau} \wedge F^{\omega}
\end{array}
$$

So on the level of cohomology, $i^* \circ d^* = \rho^* \circ (h^* \wedge 1)$ sends $U_\tau \wedge U_\omega$ to $h^*(U_\tau) \cup U_\omega$.

We now use the following theorem due to Hopf and Poincare, stated below without proof:

**Theorem 1.** *$h^*(U_\tau) = \chi(F)\mu$.*

So

$$c^* \circ i^* \circ d^*(U_\tau \wedge U_\omega) = c^*(h^*(U_\tau) \cup U_\omega) = \chi(F)v,$$

where the last equality follows by Hopf-Poincare and the definition of $\mu$.

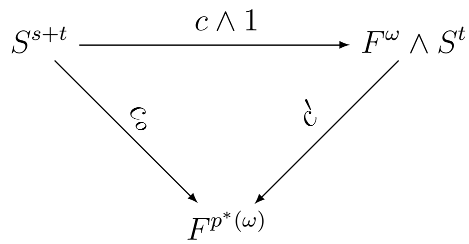
**Case 0.** *$F$ is connected and unorientable.*

Denote the orientable double cover of $F$, i.e. the set of all pairs $(x, o)$ for $x \in F$ and $o$ an orientation, by $p : X_o \to X$ given by $(x, o) \mapsto x$. Let $\omega : X_\omega \to F$ be the normal bundle of the inclusion $F \subset \mathbb{R}^s$ (by Whitney embedding), and let $F_o \subset F \times \mathbb{R}^t$ be an embedding homotopic to the projection $p$. We can identify the normal bundle, call it $X_{p*(\omega)} \to F_o$, of the composite embedding $F_o \subset \mathbb{R}^s \subset \mathbb{R}^t$ with $p^*(\omega) \times \mathbb{R}^t$. The normal bundles of these two inclusions induce the top and bottom rows of the following diagram which commutes up to homotopy:

$$
\begin{array}{ccccc}
F^{\omega} \wedge S^t & \xrightarrow{i\wedge 1} & F^{\tau\oplus\omega} \wedge S^t & & \\
\Big\uparrow{\scriptstyle p'} & & & \searrow{\scriptstyle \pi\psi\wedge 1} & \\
 & & & & S^{s+t} \\
 & & & \nearrow{\scriptstyle \pi\psi_o} & \\
F_o^{p^*(\omega)} \wedge S^t & \xrightarrow{\; i_o \;} & F_o^{p^*(\tau\oplus\omega)} & &
\end{array}
$$

where $p'$ is the projection of Thom space $F_o^{p^*(\omega)} \wedge S^t$ to $F^\omega \wedge S^t$.

The inclusions $X_{p^*(\omega)} \subset X_\omega \subset \mathbb{R}^{s+t}$ also induce the homotopy commutative triangle via the Pontryagin-collapse map:




Combining these two diagrams, we find that $(c \wedge 1)^* \circ (c')^* \circ (p')^*(i \wedge 1)^* \circ (\pi\psi \wedge 1)^* = c_o^* \circ i_o^* \circ (\pi\psi_o)^*$. It turns out (we do not present a proof here) that $(p'c')^*$ is multiplication by 2 so that the degree of $\pi\psi_o i_o c_o$ has degree twice that of $\pi\psi ic$, and because the Euler characteristic of the orientable double cover is twice that of our original space $F$ and by our results for orientable connected spaces applied to $F_o$, $\pi\psi ic$ still has degree $\chi(F)$ as desired.

**Case 0.** *$F$ is not connected.*

Say that $F$ has connected components $F_1, ..., F_m$. Then for each connected component $F_i$ we have a $\gamma_i : S^{V_i} = S^s \to (F_i^+) \wedge S^s$ defined analogously to $\gamma$ above, and $\pi \circ \gamma = \sum_i \pi \circ \gamma_i$, and we're done by the Euler characteristic's additivity over connected components. □

□

# 3 Multiplication by the Euler Characteristic

The properties of $\tau$ constructed in the previous section are enough to show the following:

**Theorem 1.** *$\tau^* \circ p^*$ is multiplication by $\chi(F)$ in singular cohomology with coefficients in any $\Lambda$.*

*Proof.* In addition to the bundle $\xi : E \xrightarrow{p} B$, we also have the bundle $\eta : F \xrightarrow{0} \{0\}$. Denote the transfer map of $\eta$ by $\tau'$. Pick some point $b \in B$, and let $i_b : \xi \to \eta$ be a bundle map covering the map $j_b$ sending $\{0\}$ to $b$.

We get from the naturality property the following diagram:

$$\begin{array}{ccccc}
H^*(S^0) & \xleftarrow{(\tau')^*} & H^*(F^+) & \xleftarrow{(p')^*} & H^*(S^0) \\
\uparrow{\scriptstyle i_b^*} & & \uparrow{\scriptstyle i_b^*} & & \uparrow{\scriptstyle i_b^*} \\
H^*(B^+) & \xleftarrow{\tau^*} & H^*(E^+) & \xleftarrow{p^*} & H^*(B^+)
\end{array}$$

Recall that $(\tau')^*(p')^*$ has degree $\chi(F)$, so starting with $1 \in H^0(B^+)$ in the bottom-right corner of the diagram, in one direction this is sent to $\tau^* p^*(1) \in H^0(B^+)$, and in the other direction this is sent to $(\tau')^*(p')^*(1) = \chi(F)$. In particular, $\tau^* p^*(1) = \chi(F) \Rightarrow \tau^*(1) = \chi(F)$.

We can then section and then extend using the following lemma:

**Lemma 1.** *Let $M$ be a ring spectrum and $N$ an $M$-module. Then for $x \in M^s(B^+)$ and $y \in N^t(E^+)$,*

$$\tau^*(p^*(x) \cup y) = x \cup \tau^*(y).$$

*Proof.* Denoting the diagonal map by $d$, we have the diagram

$$\begin{array}{ccccc}
E & \xrightarrow{d} & E \times E & \xrightarrow{p \times \mathrm{id}} & B \times E \\
\downarrow{\scriptstyle p} & & & & \downarrow{\scriptstyle 1 \times p} \\
B & & \xrightarrow{d} & & B \times B
\end{array}$$

from which we get, by naturality with respect to the bundle map $(p \times 1) \circ d$, the diagram

$$\begin{array}{ccccc}
E^+ & \xrightarrow{d} & E^+ \times E^+ & \xrightarrow{p \wedge 1} & B^+ \wedge E^+ \\
\uparrow{\scriptstyle \tau} & & & & \uparrow{\scriptstyle 1 \wedge \tau} \\
B^+ & & \xrightarrow{d} & & B^+ \wedge B^+
\end{array}$$

The commutativity of this diagram gives the desired result. □

So for $x \in \tilde{H}^s(B^+)$, we get $\tau^* \circ p^*(x) = x \cup \tau^*(1) = \chi(F) \cdot x$ by the lemma. $\square$

From now on, denote $\chi(F)$ by $\chi$. We use the fact that $p \circ \tau$ induces multiplication by $\chi$ to show the result key to proving a variant on the splitting principle for vector bundles.

**Theorem 2.** *Let $\chi \neq 0$ and let $h$ be any reduced cohomology theory on the category of finite CW complexes. The map $p^* \otimes 1 : h^*(B^+) \otimes \mathbb{Z}[\chi^{-1}] \to h^*(E^+) \otimes \mathbb{Z}[\chi^{-1}]$ is an injective homomorphism onto a direct factor.*

*Proof.* $p\tau$ induces $(p\tau)^* \otimes 1 : h^*(B^+) \otimes \mathbb{Z}[\chi^{-1}] \to h^*(B^+) \otimes \mathbb{Z}[\chi^{-1}]$.

On the $E_2$-level of the Atiyah-Herzebruch spectral sequence (unfortunately this is out of the scope of this paper, so here I use this as a black box), we have $(p\tau)^* : \tilde{H}^*(B^+; h^*(S^0) \otimes \mathbb{Z}[\chi^{-1}]) \to \tilde{H}^*(B^+; h^*(S^0) \otimes \mathbb{Z}[\chi^{-1}])$, and because this is multiplication by $\chi$, the comparison theorem tells us that $(p\tau)^* \otimes 1$ is an isomorphism so that $p^* \otimes 1$ has a left inverse as desired. $\square$

# 4 Proof of Adams Conjecture

Recall the statement of the Adams conjecture:

**Theorem 1.** *For $B$ a finite complex, $k$ any integer, and $x \in \mathrm{KO}(B)$, there is some integer $n$ for which*

$$k^n J(\psi^k(x) - x) = 0.$$

Here we will think of the $J$-homomorphism as the map $\mathrm{KO}(B) \to \mathrm{Sph}(B)$ sending a vector bundle to its associated sphere bundle.

*Proof.* Adams provided a proof for vector bundles in dimensions 1 and 2. So because we know the Adams conjecture to be true for line bundles, and because if the Adams conjecture holds for two bundles then it should hold for their direct sum, we will restrict our attention to $2n$-plane bundles.

Theorem 2 from the previous section allows us to prove a variant of the splitting principle that will allow us to go from talking about bundles of dimension $2n$, to talking about bundles associated to the principal $G$-bundle for a particular group $G$, and finally to talking about bundles of dimension 2, for which we already have a proof.

Key to this splitting principle is the notion of the wreath product, the definition of which we sketch:

**Definition 1.** *Given a set* $\Omega$*, a group* $G$ *acting on* $\Omega$*, and* $H$ *another group, the wreath product* $G \wr_\Omega H$ *is defined to be the semidirect product* $K \rtimes G$ *where* $K$ *is the direct product of copies of* $H$ *indexed by* $\Omega$*, i.e.* $\prod_{i\in\Omega} H_i$*, and acted upon by* $G$ *via* $g(h_i) = h_{g^{-1}(i)}$*.*

*When* $G = S_n$*, take* $\Omega$ *to be* $\{1, 1, ..., n\}$*.*

**Theorem 2.** *("Splitting principle") For* $\alpha : E \to B$ *a* $2n$*-plane bundle for* $B$ *a finite CW complex, there is a finite CW complex* $X$ *with a map* $\lambda : X \to B$ *and a principle* $G = S_n \wr O(2)$*-bundle* $\xi$ *over* $X$ *such that: i)* $\lambda^* : h^*(B^+) \to h^*(X^+)$ *is an injective homomorphism for any cohomology theory* $h$*, ii) the pullback* $\lambda^*(\alpha)$ *is the vector bundle associated to* $\xi$ *with fiber the "standard"* $S_n \wr O(2)$*-module* $W$*, i.e.* $\mathbb{R}^{2n}$ *which is equipped with the usual action of* $O(2n)$ *restricted to* $S_n \wr O(2)$*.*

*Proof.* Let take $T$ to be the $n$-fold product $SO(2) \times \cdots \times SO(2)$, i.e. the set of all simultaneous rotations in $n$ orthogonal 2-planes; this is a maximal torus in $O(2n)$ and also in $SO(2n)$. Let $N(T)$ denote the normalizer of $T$ in $O(2n)$, and $N'(T)$ the normalizer of $T$ in $SO(2n)$. We have a covering map $SO(2n)/T \to SO(2n)/N'(T)$ with discrete fiber $N'(T)/T$ so that $\chi(SO(2n)/T) = |N'(T)/T| \cdot \chi(SO(2n)/N'(T))$.

The following theorem allows us to conclude that $\chi(SO(2n)/N'(T)) = 1$ and, because $SO(2n)/N'(T) = O(2n)/N(T)$, that $\chi(O(2n)/N(T)) = 1$:

**Theorem 3.** *For* $G$ *a connected compact Lie group,* $T$ *a maximal torus in* $G$*,* $N(T)$ *the normalizer of* $T$ *in* $G$*,* $\chi(G/T) = |N(T)/T|$*.*

We want to use Theorem 2 by constructing $\lambda : X \to B$ with fiber $O(2n)/N(T)$, from which we would conclude property i) that $\lambda^*$ is an injection. So starting with a $2n$-plane bundle $\alpha : E \xrightarrow{p} B$, consider the associated principle $O(2n)$-bundle $\tilde{\alpha} : \tilde{E} \xrightarrow{p} \tilde{B}$ and we claim that our desired $X$ and $\lambda$ are $\tilde{E}/N(T)$ and the natural induced projection map $\tilde{E}/N(T) \to B$ so that the fiber is $O(2n)/N(T)$ as desired.

For the second part of the proof, we know that $N(T) = S_n \wr O(2)$. Observe that $E = \tilde{E} \times_{O(2n)} \mathbb{R}^{2n}$ so that the pullback $\lambda^*(\alpha)$ is $\tilde{E} \times_{N(T)} \mathbb{R}^{2n}$, but this is precisely the bundle associated with the principal $N(T)$-bundle $\xi : \tilde{E} \to \tilde{E}/N(T)$. □

For $\alpha$ a $2n$-plane bundle, the above result gives us some $\lambda : X \to B$ with the stated properties. By naturality of the $J$ homomorphism we have the diagram:

$$\begin{array}{ccc}
\mathrm{KO}(X) & \xrightarrow{J} & \mathrm{Sph}(X) \\
\uparrow{\scriptstyle \lambda^*} & & \uparrow{\scriptstyle \lambda^*} \\
\mathrm{KO}(B) & \xrightarrow{J} & \mathrm{Sph}(B)
\end{array}$$

We want to use this diagram to reduce the problem from considering $\alpha$ to considering $\lambda^*(\alpha)$, but to do this we need to show that the upward arrow on the righthand side is injective. The following steps to that end are motivated by our result that for any cohomology theory $h$, $\lambda^* : h^*(B^+) \to h^*(X^+)$ is an injective homomorphism.

We use the Stasheff's classification for fiber spaces [5], stated here without proof. Borrowing his notation, let $LF(X)$ is the set of fiberwise homotopy equivalent classes of Hurewicz fibrations $p : E \to X$ (we omit the definition of a Hurewicz fibration) with fibers homotopy equivalent to $F$, and let $[X, Y]$ denote the homotopy classes of maps $X \to Y$ (basepoint-preserving if $X$ and $Y$ are pointed).

**Theorem 4.** *For $F$ a finite CW-complex, there is a classifying space $B_H$ such that $[\cdot, B_H]$ and $LF(\cdot)$ are naturally equivalent functors.*

Let $E_n$ be the space of base-point preserving homotopy equivalences of $S^n$, and let $E$ be the colimit of these $F_n$. By taking $F$ to be $S^n$ so that $LF(B) = \mathrm{Sph}(B)$ in the theorem, we get a classifying space of $E$, call it $BE$, such that $[B^+, BE]$ and $\mathrm{Sph}(B)$ are equivalent.

To obtain an equivalence between $\mathrm{Sph}(B)$ and a cohomology theory of $B^+$, we first sketch the notion of spectra and infinite loop spaces:

**Definition 2.** *A* spectrum *is a sequence of spaces $M_0, M_1, M_2, \ldots$ such that $M_i$ is homeomorphic to the based loop space $\Omega M_{i+1}$. An* infinite loop space *is a space with homotopy type of $M_0$ of some spectrum.*

A result due to Boardman and Vogt [2] tells us that $BE$ is an infinite loop space so that $[B^+, BE]$ is equivalent to $M^0(B^+)$. $M^0$ is our desired cohomology theory so that $\lambda^* : \mathrm{Sph}(B) \to \mathrm{Sph}(X)$ is indeed injective and we can reduce to proving the Adams conjecture for $\lambda^*(\alpha)$.

Recall that $\eta = \lambda^*(\alpha) : E \times_G W \to X$ is the bundle associated with the principle $G = S_n \wr O(2)$-bundle $\xi$ with fiber $W$.

Multiplication in $G$ is given by

$$(\rho, T_1, ..., T_n) \cdot (\sigma, S_1, ..., S_n) = (\rho\sigma, T_{\sigma(1)}S_1, ..., T_{\sigma(n)}S_n).$$

Let $H$ be the subgroup of $G$ elements $(\rho, T_1, ..., T_n)$ such that $\rho(1) = 1$. This way, we can define an $H$-action $H \to O(2)$ sending $(\rho, T_1, ..., T_n)$ to $T_1$, giving a 2-dimensional $H$-module that we will call $V$.

$H$ has index $n$ inside $G$; denote its cosets by $\sigma_1 H, ..., \sigma_n H$ and define $i(V)$ to be $\bigoplus_{i=1}^{n} \{\sigma_i\} \times V$. Define the action of $G$ on $i(V)$ by $g(\sigma_i, v) = (\sigma_k, hv)$, where $g\sigma_i = \sigma_k h$.

Recalling that $\alpha$ was defined to be the bundle with fiber $\mathbb{R}^{2n}$associated to the principle $O(2n)$-bundle $\tilde{E} \to B$, we get a covering space

$$\tilde{E}/H \to \tilde{E}/G$$

with discrete fiber of cardinality $n$. Also recall that $X = \tilde{E}/G$ in the construction for Theorem 2.

We have a transfer map $\tau^* : \mathrm{KO}(\tilde{E}/H) \to \mathrm{KO}(X)$, and this sends the bundle with fiber $V$, an $H$-module, associated with $\tilde{E} \to \tilde{E}/H$ to the bundle with fiber $i(V)$, a $G$-module, associated with $\tilde{E} \to \tilde{E}/G$. But as a $2n$-dimensional $G$-module, $i(V)$ is just $W$, so $\tau^*$ sends the two-dimensional vector bundle $\zeta : \tilde{E} \times_H V \to \tilde{E}/H$ to $\eta$.

By Adams' proof of the conjecture for bundles of dimension 2, we are done once we prove the following claim:

**Claim 1.** *For $\zeta$ any two-dimensional vector bundle over $\tilde{E}/H$, the Adams conjecture holds for $\tau^*(\zeta)$.*

*Proof.* Quillen's proof in the case of $k$ an odd prime extends immediately to the cases of $k$ odd and of $k$ even and $\zeta$ orientable (we refer the reader to [4]).

For the case of $k$ even and $\zeta$ non-orientable, tensoring $\zeta$ with the line bundle $\lambda$ whose first Stiefel-Whitney class is $w_1(\zeta)$ kills the obstruction to orientability. $\zeta \otimes \lambda$ is a two-dimensional orientable bundle and thus $\tau^*(\zeta \otimes \lambda)$ we already know satisfies the Adams conjecture.

But $2^m (\tau^*[\zeta \otimes \lambda] - \tau^*[\lambda]) = 0$ for some $m$ because $2^m([\lambda] - 1) = 0$, and because $k$ is even, we certainly have $k^n J(\psi^k(\tau^*[\lambda]) - \tau^*[\lambda]) = 0$. □

□

# References


[1] J. C. Becker, D. H. Gottlieb: The transfer map and fiber bundles, *Topology* 14 (1975), 1-12.

[2] J. M. Boardman and R. M. Vogt, Homotopy everything $H$-spaces, *Bull. Am. math. Soc.* 74 (1968), 1117-1122.

[3] Miller, Haynes. "Vector Fields on Spheres, etc." (2010).

[4] D. Quillen: The Adams conjecture, *Topology* 10 (1971), 67-80.

[5] J. D. Stasheff: A classification theorem for fiber spaces, *Topology* 2 (1963), 239-246.